Winter term 2024/25
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Presence sheet 02

Mathematics for Machine Learning

Tutorial of **Week 03 (28.10. - 01.11.2024)**

Exercise 1 (**Rank-Nullity Theorem).**
Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} \tag{1}$$

which describes a linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(x) = Ax$.

a) Determine the rank of A.

b) Determine the dimension of the kernel of A.

c) Determine a basis of the image and a basis of the kernel of A.

Exercise 2 (**Rank and Inverse).**
Consider the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2}$$

a) Is A invertible?

b) If yes, calculate the inverse A^{-1}.

Exercise 3 (**Determinant).**
Calculate the determinant of the following matrices

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

b) $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

c) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 8 & 9 \end{pmatrix}$.

Exercise 4 (**Rotation matrix).**

a) Consider the rotation matrix $A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$.
Show that A has no real eigenvalues. Try to understand where the name "rotation matrix" comes from!

b) Find a (non-diagonal) matrix $A \in \mathbb{R}^{2\times 2}$ with exactly one (real) eigenvalue.

Exercise 5 (**Eigenvectors).**
Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

a) Find one eigenvector of A for eigenvalue $\lambda_1 = 3$.

b) Find one eigenvector of A for eigenvalue $\lambda_2 = 1$.

c) Are there more than one eigenvector for each eigenvalue?

d) Are there more eigenvalues?

Exercise 6 (**Eigenspaces).**
Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

a) Calculate the eigenspace of A for eigenvalue $\lambda_1 = 1$, that is, $E(1, A)$.

b) Calculate the eigenspace of A for eigenvalue $\lambda_2 = 2$, that is, $E(2, A)$.

c) Find a different basis of $E(1, A)$ and $E(2, A)$.

d) What is the geometric multiplicity of λ_1 and λ_2

Exercise 7 (**Inverse matrices).**
Assume you take “just any” 3x3 matrix. Intuitively, what do you think is more “likely”: that it is invertible or not? Discuss informally.