Winter term 2020/21
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Assignment 5

Mathematics for Machine Learning

Submission due on **07.12.20, 8:00**

Justify all your claims.

Exercise 1 (**Positive (semi-)definite matrices**, 3+2+3 points)**.**

a) Are the following matrices positive semi-definite? Are they positive definite?

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{for } \theta \in \mathbb{R}$$

$$C = \begin{pmatrix} 2 & 4 & 1 & 0 \\ 0 & -3 & 1 & -7 \\ 0 & 0 & 12 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 1 & 1 & 12 & 0 \\ 0 & -7 & 3 & 1 \end{pmatrix}$$

b) Let $A \in \mathbb{C}^{n\times n}$ be hermitian and positive (semi-)definite. Prove that A^k is symmetric and positive (semi-)definite as well.

c) Consider a symmetric and positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$. Assume additionally that there exists no vector $x \in \mathbb{R}^n \setminus \{0\}$ which satisfies $x^t A y = 0$ for all $y \in \mathbb{R}^n$. Prove that A is positive definite.

Exercise 2 (**Singular value decomposition**, 2+1+2+2 points)**.**

a) Compute the singular values of $A = \begin{pmatrix} -\frac{1}{2} & \frac{3\sqrt{3}}{2} \\ 0 & 0 \\ \frac{3\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix} \in \mathbb{R}^{3\times 2}$.

Hint: Proceed as in the proof sketch for the proposition about the SVD in lecture 29 (Singular value decomposition).

b) Prove that $A \in \mathbb{R}^{m\times n}$ and $A^t \in \mathbb{R}^{n\times m}$ have the same singular values.

c) Prove that the Frobenius norm defined by $\|A\|_F = \sqrt{\operatorname{tr}(A^t A)}$ can be computed as $\|A\|_F = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}$, where σ_i are the singular values of $A \in \mathbb{R}^{m\times n}$.

d) Consider a matrix $A \in \mathbb{R}^{m\times n}$ with singular value decomposition $A = U\Sigma V^t$, where $U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$ are orthonormal and $\Sigma \in \mathbb{R}^{m\times n}$ is diagonal. The lecture stated that the matrix $A^\# = V\Sigma^\# U^t$ is a pseudo-inverse of A, where $\Sigma^\# \in \mathbb{R}^{n\times n}$ is obtained from Σ by transposing and inverting every non-zero element. Prove this statement.

Exercise 3 (**Spectral clustering**, 1+1+1+2 points)**.** Let $W \in \mathbb{R}^{n\times n}$ be a symmetric matrix with non-negative entries and $D \in \mathbb{R}^{n\times n}$ the diagonal matrix which contains the row sums of W, that is, $d_{i,i} = \sum_{j=1}^n w_{i,j}$. Define $L := D - W$.

a) Prove that for all $x \in \mathbb{R}^n$ it holds

$$x^t L x = \frac{1}{2} \sum_{i,j=1}^{n} w_{i,j} (x_i - x_j)^2 \,.$$

b) Conclude that L is symmetric and positive semi-definite.

c) Show that the vector of constant ones $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$ is an eigenvector of L.

d) Solve the constrained minimization problem

$$\min_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} x^t L x \quad \text{subject to } \langle x, \mathbb{1} \rangle = 0 \,.$$