Winter term 2020/21
L. Rendsburg / U. von Luxburg

Assignment 4

Mathematics for Machine Learning

Submission due on **30.11.20, 8:00**

Justify all your claims.

Exercise 1 (**Scalar product**, 2+2+2 points). Prove that the following functions $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ define a scalar product.

a) $V = \mathbb{C}^n$ with $\langle x, y \rangle = \sum_{j=1}^n x_j \overline{y}_j$ for $x, y \in \mathbb{C}^n$.

b) $V = \mathbb{R}^{n \times n}$ with $\langle A, B \rangle = \mathrm{tr}(A^t B)$ for $A, B \in \mathbb{R}^{n \times n}$.

Let $(V, \langle \cdot, \cdot \rangle_0)$ be a scalar product space over $\mathbb{R}$ and define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, (v, w) \mapsto \langle v, w \rangle = \langle Sv, Sw \rangle_0$ for an $S \in \mathcal{L}(V)$.

c) Prove that $\langle \cdot, \cdot \rangle$ is a scalar product if and only if S is injective.

Exercise 2 (**Scalar products and norms**, 2+2+1 points). Consider a normed vector space $(V, \|\cdot\|)$. The parallelogram equality states that the norm is given by a scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ via $\|v\| = \sqrt{\langle v, v \rangle}$ if and only if for all $v, w \in V$ it holds that

$$\|v + w\|^2 + \|v - w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right) \tag{1}$$

a) Prove the forward implication. That is, prove that Eq. (1) holds if the norm is given by a scalar product.

b) Let $p > 0$. Prove that there is a scalar product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|x\| = (|x_1|^p + |x_2|^p)^{1/p}$$

for all $x \in \mathbb{R}^2$ if and only if $p = 2$.

c) Let $(V, \|\cdot\|)$ be a normed vector space, where the norm is given by a scalar product. Prove the Pythagorean theorem $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ for all orthogonal $u, v \in V$.

Exercise 3 (**Cauchy-Schwarz inequality**, 2+1+1 points). Let $(V, \langle \cdot, \cdot \rangle)$ be a scalar product space. The Cauchy-Schwarz inequality states that all $u, v \in V$ satisfy

$$|\langle u, v \rangle| \leq \|u\| \, \|v\| \; .$$

a) Prove that $16 \leq (a + b + c + d)(1/a + 1/b + 1/c + 1/d)$ for all $a, b, c, d > 0$.

b) Prove that $(x_1 + \cdots + x_n)^2 \leq n\left(x_1^2 + \cdots + x_n^2\right)$ for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$.

c) Prove that for continuous real-valued functions f, g on $[a, b] \subseteq \mathbb{R}$ with $a < b$ it holds that $\left|\int_a^b f(x)g(x)dx\right|^2 \leq \left(\int_a^b f(x)^2 dx\right)\left(\int_a^b g(x)^2 dx\right)$.

Exercise 4 (**Gram–Schmidt orthonormalization**, 3+2 points). The Gram-Schmidt orthonormalization takes any basis $v_1, \ldots, v_n$ of a scalar product space $(V, \langle \cdot, \cdot \rangle)$ and outputs an orthonormal basis $u_1, \ldots, u_n$. It starts with $u_1 = v_1 / \|v_1\|$ and then iteratively applies the update

$$\begin{aligned} u'_{k+1} &= v_{k+1} - P_{U_k}(v_{k+1}) = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1}, u_i \rangle \, u_i \\ u_{k+1} &= \frac{u'_{k+1}}{\|u'_{k+1}\|}, \end{aligned}$$

for $k = 1, \ldots, n-1$, where $U_k = \operatorname{span}(u_1, \ldots, u_k)$.

a) Consider the subspace $V = \operatorname{span}\left(1, x, x^2, x^3\right) \subseteq \mathbb{R}^{\mathbb{R}}$ of polynomials with degree ≤ 3 with the scalar product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$ for $f, g \in V$. Apply Gram-Schmidt to the basis $1, x, x^2, x^3$ in order to compute an orthonormal basis of V.
Hint: You may use a computer to compute any integrals. To check whether your solution is correct simply verify whether your basis is orthonormal.

b) What happens if the procedure is applied to a list of vectors $(v_1, \ldots, v_m)$ that is not linearly independent?