Winter term 2020/21
L. Rendsburg / U. von Luxburg

Assignment 2

Mathematics for Machine Learning

Submission due on **16.11.20, 8:00**

Exercise 1 (**Eigenvalues and eigenspaces**, 2+1+2+1 points)**.** Consider the matrices

$$A = \begin{pmatrix} -8 & 2 & 16 \\ 0 & -2 & 0 \\ -3 & 1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Compute the characteristic polynomials p_A, p_B, and p_C.

b) Compute the eigenvalues of A, B, and C over $\mathbb{R}$.

c) For every matrix and eigenvalue, give a basis of the corresponding eigenspace.

d) What are the algebraic and geometric multiplicities of the eigenvalues?

Exercise 2 (**Matrices I**, 1+1+1+1 points)**.** Consider the differentiation operator $D = \mathrm{d}/\mathrm{d}t\colon \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}, f \mapsto f'$ on the vector space $\mathbb{R}^{\mathbb{R}}$ of all real functions. In the following cases we give bases $\mathcal{W}$ with corresponding subspaces $\mathcal{U} = \operatorname{span}(\mathcal{W})$, such that D restricted to $\mathcal{U}$ describes a linear map $D|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{U}$. Note that $\operatorname{range}(D|_{\mathcal{U}}) \subseteq \mathcal{U}$ is an additional statement and does not hold for general $\mathcal{U}$. State the matrix $\mathcal{M}(D|_{\mathcal{U}})$ with respect to the corresponding basis $\mathcal{W}$ in each of the cases.

a) $\mathcal{W} = \left(e^t, e^{2t}\right)$

b) $\mathcal{W} = \left(1, t, t^2, t^3, t^4\right)$

c) $\mathcal{W} = \left(e^t, te^t\right)$

d) $\mathcal{W} = (\sin t, \cos t)$

Exercise 3 (**Matrices II**, 1+2+2 points)**.**

a) Give an example for a 2-by-2 matrix A with $A \neq 0$, but $A^2 = 0$.

b) Consider two square matrices $A, B \in \mathbb{R}^{n\times n}$ with $A \cdot B = I$. Prove that $B \cdot A = I$

c) Consider a matrix $A \in \mathbb{R}^{n\times n}$ with $A^2 = 0$. Show that $A - I$ is invertible.

Exercise 4 (**Eigenvalues I**, 2+3 points)**.**

a) Let $A \in \mathbb{R}^{n\times n}$ with $A^k = 0$ for some $k \in \mathbb{N}$. Prove that if λ is an eigenvalue of A, then $\lambda = 0$.

b) Let V be a finite-dimensional vector space and $T\colon V \to V$ a linear map such that every $v \in V$ with $v \neq 0$ is an eigenvector of T. Prove that $T = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{R}$.

Bonus exercise (**Eigenvalues II**, 3+1 points)**.** Consider a matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalue $\lambda \in \mathbb{R}$.

a) Consider a polynomial $p(x) = \sum_{k=0}^{K} c_k x^k$ with $c_i \in \mathbb{R}$. Show that $p(\lambda) = \sum_{k=0}^{K} c_k \lambda^k$ is an eigenvalue of $p(A) = \sum_{k=0}^{K} c_k A^k$, where $A^0 = I$.

b) Assume that A is invertible and $\lambda \neq 0$. Show that $1/\lambda$ is an eigenvalue of A^{-1}.