Winter term 2020/21
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Assignment 1

Mathematics for Machine Learning

Submission due on **09.11.20, 8:00**

Exercise 1 (**Groups and vector spaces**, 6+2+1 points)**.**

a) Decide whether the following statements are true. Justify your answer formally.

   i) $(\mathbb{R}, \Box)$ is a group, where $\Box$ denotes the mean between two elements, that is, $x \Box y = (x+y)/2$ for $x, y \in \mathbb{R}$.

   ii) The set of functions $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ is a vector space over $\mathbb{R}$ with addition $(f+g)(x) = f(x) + g(x)$ and scalar multiplication $(\lambda f)(x) = \lambda f(x)$ for $x, \lambda \in \mathbb{R}$, $f, g \in \mathcal{F}$.

   iii) The empty set $V = \emptyset$ is a vector space over $\mathbb{R}$.

   iv) For any finite set X, the power set $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is a group with the symmetric difference $\triangle$, that is, $A \triangle B = (A \setminus B) \cup (B \setminus A)$ for $A, B \in \mathcal{P}(X)$.
   Hint: You may use a Venn diagram (a visualization of three sets drawn as circles, which shows all possible intersections) to reason about equations.

b) Let $(G, \circ)$ be a group. Prove that the neutral element and inverse elements are unique.

c) Let $(G, \circ)$ be a group with neutral element $e \in G$ such that $x \circ x = e$ for all $x \in G$. Prove that G is commutative.

Exercise 2 (**Subspaces and dimension**, 3+2 points)**.** Let $U_1, \ldots, U_m$ be subspaces of a finite-dimensional vector space.

a) Assume that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and that

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m \,. \tag{1}$$

b) It holds that $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. Similar to the inclusion-exclusion principle, which states that for three sets A, B, C it holds

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \,,$$

one might guess that

$$\begin{aligned} \dim(U_1 + U_2 + U_3) = &\dim U_1 + \dim U_2 + \dim U_3 \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3) \,. \end{aligned} \tag{2}$$

Prove Eq. (2) or give a counterexample.

Exercise 3 (**Linear maps**, 2+2+2 points)**.**

a) Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\operatorname{range} T = \ker T$.

b) Prove that there does not exist a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ such that $\operatorname{range} T = \ker T$.

c) Suppose U is a subspace of V with $U \neq V$, and let $S\colon \mathcal{L}(U, W)$ be a linear map to another vector space W, such that $S \neq 0$ (there exists an element $u \in U$ with $Su \neq 0$). Define $T\colon V \to W$ by extending S with zeros, that is,

$$Tv = \begin{cases} Sv & \text{if } v \in U\,, \\ 0 & \text{if } v \in V \setminus U\,. \end{cases}$$

Prove that T is not a linear map on V.