Winter term 2020/21
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Assignment 7

Mathematics for Machine Learning

Submission due on **21.12.20, 8:00**
Assignment 8 will be released on 10.01.21 and is due on 17.01.21.

Justify all your claims.

Exercise 1 (**Measures**, 1+1+3+3 points)**.**

a) Consider the set $X = \{1, 2, 3, 4\}$. Is the following set $\mathcal{F} \subset \mathcal{P}(X)$ a σ-algebra on X?

$$\mathcal{F} = \{\emptyset, \{3\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$$

b) Let X be a non-empty set and $A \subseteq X$. Compute the σ-algebra $\sigma(\{A\})$ generated by A.

c) Let X be an uncountable set. Consider the collection of sets $\mathcal{F} = \{A \subset X \mid A \text{ or } A^c \text{ countable}\}$ and the function $\mu \colon \mathcal{F} \to \{0, 1\}, A \mapsto \begin{cases} 0, & \text{if } A \text{ countable} \\ 1, & \text{if } A^c \text{ countable} \end{cases}$, where $A^c := X \setminus A$ denotes the complement. Prove that $(X, \mathcal{F}, \mu)$ is a measure space. In particular, prove that μ is well-defined.
Hint: You may use the fact that countable unions of countable sets are countable.

d) Consider a measure space $(X, \mathcal{F}, \mu)$. Prove the following properties:

i) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{i \in \mathbb{N}} A_i \in \mathcal{F}$.

ii) If $A_1, \ldots, A_m \in \mathcal{F}$, then $\bigcup_{i=1}^m A_i \in \mathcal{F}$. If the A_i are additionally disjoint, then $\mu(\bigcup_{i=1}^m A_i) = \sum_{i=1}^m \mu(A_i)$.
Hint: Show that $\emptyset \in \mathcal{F}$.

iii) For $A, B \in \mathcal{F}$, it holds $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

Exercise 2 (**Derivatives**, 4+2+1 points)**.**

a) Compute the Jacobian matrix of the following functions

$$f_1 \colon \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} \cos t \\ t^2 - 2 \end{pmatrix} \qquad f_2 \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \begin{pmatrix} x_1^3 - 3x_1x_2^2 \\ 3x_1^2x_2 - x_2^3 \end{pmatrix}$$

$$f_3 \colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto x^t A x \quad \text{for } A \in \mathbb{R}^{n \times n} \qquad f_4 \colon \mathbb{R}^{n \times m} \to \mathbb{R}, \quad X \mapsto a^t X b \quad \text{for } a, b \in \mathbb{R}^n$$

b) Prove that the function $f \colon \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, X \mapsto X^2$ is differentiable and that the Jacobian matrix $Df(X)$ at $X \in \mathbb{R}^{n \times n}$ satisfies

$$Df(X)(H) = HX + XH \quad \text{for } H \in \mathbb{R}^{n \times n}.$$

Hint: By definition, $Df(X) \colon \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is given via $(Df(X)(H))_{i,j} = \sum_{k,l=1}^n \partial_{X_{k,l}} f_{i,j}(X) H_{k,l}$ for $i, j \in \{1, \ldots, n\}$ and $H \in \mathbb{R}^{n \times n}$.

c) Consider the function $g \colon \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}, (x, y, z) \mapsto \frac{\cos(xy)}{z^2}$. Compute the directional derivative of g at $\xi = (1, \pi/2, -3)$ in the direction $v = (1, 2, 3)$. What is the direction of steepest descent (the direction for which the directional derivative attains its smallest value) at ξ?

Exercise 3 (**Extremal points**, 2+1+2 points). Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x^3 + 1/3y^3 - 12x - y$.

a) Compute the set of critical points for f and classify them into local minima, local maxima, or saddle point.

b) Does f have a global minimum or global maximum?

c) Consider the function $g\colon \mathbb{R}^3 \to \mathbb{R}, (x,y,z) \mapsto \alpha x^2 e^y + y^2 e^z + z^2 e^x$ with $\alpha \in \mathbb{R}$. For which values of α is $(0,0,0)$ a local minimum, local maximum, or saddle point?

Bonus exercise (**More measure theory**, 2+3 points). Consider a measure space $(X, \mathcal{F})$, where $\mathcal{F} = \sigma(\mathcal{E})$ is generated by $\mathcal{E} \subseteq \mathcal{P}(X)$. To show that all elements in $\mathcal{F}$ have a property E, we can use the following principle: first, we show that the collection $\mathcal{G} = \{A \subset X \mid A \text{ has property } E\}$ of sets with property E is a σ-algebra. Next, we show that it contains the generator $\mathcal{E}$, that is, $\mathcal{E} \subseteq \mathcal{G}$.

a) Argue how the above ($\mathcal{G}$ is a σ-algebra with $\mathcal{E} \subseteq \mathcal{G}$) can be used to conclude that all elements in $\mathcal{F}$ have property E.

b) Consider a map $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ between two measurable spaces, where $\mathcal{B} = \sigma(\mathcal{E})$ for a collection of subsets $\mathcal{E} \subseteq \mathcal{P}(Y)$. Use the above principle to prove that f is measurable if and only if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{E}$.

We wish you happy holidays!