Winter term 2020/21
L. Rendsburg / U. von Luxburg

Assignment 6

Mathematics for Machine Learning

Submission due on **14.12.20, 8:00**

Justify all your claims.

Exercise 1 (**Continuous functions**, 2+2 points)**.**

a) Consider the normed spaces $\left(L_p([0,1]), \|\cdot\|_p\right)$ and $\left(\mathbb{R}^n, \|\cdot\|_q\right)$ with $n \in \mathbb{N}$ and $p, q \in [1, \infty]$, where $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)|$ for $f \in L_\infty([0,1])$. Let $[\delta_x] \in L_p([0,1])$ be the equivalence class of the indicator function $\delta_x(y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{else} \end{cases}$ for $y \in [0,1]$. Prove whether the following functions are continuous:

$$f\colon \left(\mathbb{R}^3, \|\cdot\|_2\right) \to \left(\mathbb{R}^2, \|\cdot\|_1\right), \quad (x, y, z) \mapsto (x + 2y, z^2)$$

$$g\colon (\mathbb{R}, \|\cdot\|_1) \to (\mathbb{R}, \|\cdot\|_1), \quad x \mapsto \operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{otherwise} \end{cases}$$

$$h_1\colon \left([0,1], \|\cdot\|_q\right) \to \left(L_p([0,1]), \|\cdot\|_p\right), \quad x \mapsto [\delta_x] \quad \text{with } q \in [1,\infty] \text{ and } p \in [1,\infty)$$

$$h_2\colon \left([0,1], \|\cdot\|_q\right) \to \left(L_\infty([0,1]), \|\cdot\|_\infty\right), \quad x \mapsto [\delta_x] \quad \text{with } q \in [1,\infty]$$

b) Let $(V, \|\cdot\|)$ be a normed vector space and $\|\cdot\|_a, \|\cdot\|_b$ two norms on $\mathbb{R}^n$ with $n \in \mathbb{N}$. Prove that every continuous map $f\colon (V, \|\cdot\|) \to (\mathbb{R}^n, \|\cdot\|_a)$ is also continuous when the norm on $\mathbb{R}^n$ is replaced by $\|\cdot\|_b$.

Exercise 2 (**Supremum and infimum**, 1+1+2 points)**.**

a) Compute the lim inf and lim sup of the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = (-1)^n(1 + 1/n)$.

Consider two sets $A \subseteq B \subset \mathbb{R}$.

b) Prove that $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

c) Prove that $\inf A = -\sup(-A)$, where $-A = \{-a \ \in A\}$.

Exercise 3 (**Uniform convergence**, 2+1+1+2 points)**.**

a) Prove whether the following sequences of functions converge pointwise. If they do state the limit and prove whether they converge uniformly.

$$(f_n)_{n\in\mathbb{N}} \quad \text{with} \quad f_n\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{n}\sin(nx)$$

$$(g_n)_{n\in\mathbb{N}} \quad \text{with} \quad g_n\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + \frac{x}{n}\cos(x)$$

b) Consider a sequence of functions $f_n\colon \mathcal{D} \to \mathbb{R}$ on a finite set $\mathcal{D}$ that converges pointwise to a function $f\colon \mathcal{D} \to \mathbb{R}$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Consider a sequence of functions $f_n \colon [a,b] \to \mathbb{R}$, which are all Lipschitz continuous with the same Lipschitz constant $L > 0$. Assume that this sequence converges pointwise to $f \colon [a,b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$ and $a < b$.

c) Prove that f is also Lipschitz continuous with Lipschitz constant L.

d) Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.
Hint: For an $\varepsilon > 0$ consider the set $\mathcal{D}$ of points $a, a+\varepsilon, a+2\varepsilon, \ldots$ up to b. Then let $\lfloor x \rfloor := \max\{y \in \mathcal{D} \mid y \leq x\}$ and use the equality

$$f_n(x) - f(x) = f_n(x) - f_n(\lfloor x \rfloor) + f_n(\lfloor x \rfloor) - f(\lfloor x \rfloor) + f(\lfloor x \rfloor) - f(x)$$

Exercise 4 (Power and Taylor series, 2+2+2 points**).**

a) Determine the convergence radius of the following power series:

$$\sum_{j=1}^{\infty} \frac{j^2}{2^j} x^j \quad \text{and} \quad \sum_{j=1}^{\infty} 3^j x^{j^2}$$

b) Compute the Taylor series of $f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto e^{\pi - x} \sin x$ in $a = \pi$ up to degree $n = 3$ with the Lagrange remainder. Find an upper bound for the remainder by bounding $\sup_{\xi \geq 0} f^{(4)}(\xi)$ with a constant.

c) Prove that f from part b) is equal to its Taylor series, that is, $f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} T_k(x, \pi)$ for all $x \in \mathbb{R}_{\geq 0}$.
Hint: What is the connection between f and $f^{(4)}$?