Assignment 3

Mathematics for Machine Learning

Submission due on **23.11.20, 8:00**

Justify all your claims.

Exercise 1 (**Diagonalizable matrices**, 2+2+2+1 points)**.**

a) Consider the matrix $A \in \mathbb{R}^{3\times 3}$ given by its representation as a diagonal matrix

$$A = P^{-1}DP = \begin{pmatrix} 1 & -1 & 1 \\ 0.5 & 0 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & -1 \end{pmatrix},$$

where P describes the change of basis. What are the eigenvalues of A? Is A invertible? Verify that the trace of A equals the sum of its eigenvalues.

b) Compute a basis for each eigenspace of A. What is the relationship between their basis vectors and P^{-1}?

c) Consider a general matrix $A \in \mathbb{R}^{n\times n}$ with decomposition $A = P^{-1}DP$ into an invertible matrix $P \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{n\times n}$. Derive an explicit formula for the matrix power A^k with $k \in \mathbb{N}$ based on this decomposition.

d) Assume additionally that the entries on the diagonal of D are non-zero. Derive a formula for the inverse A^{-1} based on the decomposition.

Exercise 2 (**Topology**, 3+1+2+1 points)**.** Justify all your claims formally for the following exercises.

a) Decide whether the following two sets are open and whether they are closed.

$$S_1 = B_r(0) = \{x \in \mathbb{R}^n \mid \|x\|_2 < r\} \quad \text{for } r > 0$$
$$S_2 = \mathbb{R}^n$$

Hint: You may use the fact that the metric space $(\mathbb{R}^n, \|\cdot\|_2)$ is complete.

b) For a real matrix $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^m$, decide whether the following set is convex in general. If it is not, give a counterexample.

$$S_3 = \{x \in \mathbb{R}^n \mid Ax = b\}$$

c) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *convex*, if it satisfies $f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$ for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$. Decide whether the following two sets are convex for all convex functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and choices of $c \in \mathbb{R}$. If they are not, give a counterexample.

$$S_4 = \{f \le c\} = \{x \in \mathbb{R}^n \mid f(x) \le c\}$$
$$S_5 = \{f > c\} = \{x \in \mathbb{R}^n \mid f(x) > c\}$$

d) Prove that if $C, D \subseteq \mathbb{R}^n$ are convex sets, then the set sum $C + D = \{x+y \mid x \in C, y \in D\}$ is also convex.

Exercise 3 (**Norms and metrics**, 2+2+2 points)**.**

a) The Hamming space $\{0,1\}^n$ is the set of all binary lists of length n. The Hamming distance d counts the number of distinct components, that is,

$$d\colon \{0,1\}^n \times \{0,1\}^n \to \mathbb{N}_0,$$
$$(x,y) \mapsto d(x,y) = \sum_{i=1}^{n} \mathbb{1}_{\{x_i \neq y_i\}}\,,$$

where $\mathbb{1}_{\{x_i \neq y_i\}} = \begin{cases} 1, & \text{if } x_i \neq y_i \\ 0, & \text{otherwise} \end{cases}$. Prove that $(\{0,1\}^n, d)$ is a metric space.

b) Consider the p-norm $\|x\|_p = \left(\sum_{k=1}^{d} |x_k|^p\right)^{1/p}$ on $\mathbb{R}^n$. Prove that $\|\cdot\|_p$ is not a norm for $0 < p < 1$.

c) Show that every norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ induces a metric via $d(x,y) = \|x - y\|$.

Bonus exercise (**Convex sets induce norms**, 3+2 points)**.** Consider the set of points inside an ellipse

$$C = \left\{ x \in \mathbb{R}^2 \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1 \right\} \subseteq \mathbb{R}^2$$

with $a, b > 0$. C is closed and has non-empty interior.

a) Verify the remaining conditions for the theorem given in lecture 19 "Convex set induces a norm", that is: C is symmetric, bounded, and convex.

b) Derive an explicit formula for the norm induced by C.