Winter term 2020/21
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Assignment 13 - Test exam

Mathematics for Machine Learning

Submission due on **22.02.21, 8:00**

Justify all your claims.

The point system on this sheet is similar to the point system in the exams. There are 54 points in total, which corresponds to 20 bonus points for your assignments.

We would roughly suggest you try to solve it in 110 minutes.

Every multiple-choice question has exactly one correct answer. Every correctly answered multiple-choice question is worth 1 point. **You do not loose any points for wrong answers on multiple-choice questions. Therefore, make sure to answer all of them!**

Good Luck!

Exercise 1 (**Linear algebra - Multiple choice**, 3 points)**.**

1. Let $A = uv^T$ for $u, v \in \mathbb{R}^n$ with $u, v \neq 0$. Then it holds $\text{rank}(A) = 2$.

   A. True B. False

2. An orthonormal basis of a vector space is unique up to ordering of the basis vectors.

   A. True B. False

3. Let $C = A \cdot B$ with $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$. In which of these cases is C *not* invertible for any possible A, B?

   A. $n > m$ B. $n < m$ C. $n = m$ D. There always exist A, B s.th. C is invertible

Exercise 2 (**Linear algebra 1**, 5 points)**.**

Find the eigenvalues and one eigenvector each for the matrix $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Is A invertible? Justify your answer!

Exercise 3 (**Linear algebra 2**, 5 points)**.**
Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Prove that $\mathrm{tr}(A)$ is equal to the sum of the eigenvalues of the matrix.

Exercise 4 (**Linear algebra 3**, 5 points)**.**
Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. We denote their eigenvalues as $\lambda_i(A)$, with $i \in 1, \dots, n$ and the eigenvalues sorted in ascending order (i.e. $\lambda_1(A) \leq \lambda_2(A) \leq \dots \leq \lambda_n(A)$) (analogous for B). Prove that

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B)$$

Hint: Use the Courant-Fischer-Weyl theorem:

$$\lambda_k(A) = \min_{\substack{U \text{ subspace,} \\ \dim U = k}} \max_{x \in U\setminus\{0\}} R_A(x)$$

with the Rayleigh-Quotient $R_A(x) = \frac{\langle x, Ax\rangle}{\langle x, x\rangle}$.

Exercise 5 (**Calculus - Multiple choice**, 3 points)**.**

1. For every fixed $x \in \mathbb{R}$, the function $f \colon \mathbb{R} \to \mathbb{R}, z \mapsto |x|^z$ is differentiable.

   A. True

   B. False

2. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a polynomial. Then the Taylor series of f at any point $x_0 \in \mathbb{R}$ coincides with f everywhere, that is, the radius of convergence is $r = \infty$.

   A. True

   B. False

3. Consider two finite sets $A, B \subseteq \mathbb{R}$. Then it holds $\max(A + B) = \max A + \max B$, where $A + B \coloneqq \{a + b \mid a \in A, b \in B\}$.

   A. True

   B. False

Exercise 6 (**Calculus 1**, 5 points)**.**
Consider a sequence of continuous functions $f_1, f_2, \ldots : \mathbb{R} \to \mathbb{R}$ with $f_n \to f$ uniformly for some function $f \colon \mathbb{R} \to \mathbb{R}$. Additionally, consider a sequence of points $x_1, x_2, \ldots \in \mathbb{R}$ with $x_n \to x$ for some $x \in \mathbb{R}$. Prove that $f_n(x_n) \to f(x)$.
Hint: Use the decomposition $f_n(x_n) - f(x) = f_n(x_n) - f(x_n) + f(x_n) - f(x)$.

Exercise 7 (**Calculus 2**, 5 points)**.**
Consider a measure space $(\Omega, \mathcal{F}, \mu)$ and a sequence of increasing sets $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{F}$. Prove that $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n)$.
Hint: Rephrase the terms by using the sequence of disjoint increments $(B_j)_{j\in\mathbb{N}}$, defined by $B_j = A_j \setminus A_{j-1}$ (where $A_0 \coloneqq \emptyset$).

Exercise 8 (**Probability theory/ Statistics - Multiple choice**, 3 points).

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A, B \in \mathcal{F}$ with $P(A) = P(B) = 1$. Then it holds $P(A \cap B) = 1$.

   A. True B. False

2. Let X, Y be two random variables that are uniformly distributed on $[0, 1]$. Then X and Y are independent.

   A. True B. False

3. Consider a parametric family $\mathcal{F} = \{f_\theta \mid \theta \in \Theta\}$ of distributions on $\mathbb{R}^d$. For fixed $x \in \mathbb{R}^d$ and $\theta \in \Theta$, the likelihood function is $\mathcal{L}(\theta) = f_\theta(x)$. Then it holds $\int_{\theta \in \Theta} \mathcal{L}(\theta) d\theta = 1$.

   A. True B. False

Exercise 9 (Probability theory/ Statistics 1, 5 points**).**
A traffic light is red for 30 seconds an green for 30 seconds. A car arrives at a random point in time. Give a suitable probability space that describes this random experiment. What is the expected waiting time?

Exercise 10 (**Probability theory/ Statistics 2**, 5 points)**.**
Let X be a discrete random variable on $\mathcal{X} = \{x_n \in [0,1] \mid n \in \mathbb{N}\}$ and let Y be an independent, uniformly distributed random variable on $[0,1]$. Prove that

$$P(Y \leq X) = \mathbb{E}\left[X\right] .$$

Exercise 11 (**Probability theory/ Statistics 3**, 5 points)**.**
Let $X_1, X_2, \ldots$ be random variables on $\mathbb{R}$ with $P(X_n = 2^n) = 1/2^n$ and $P(X_n = 0) = 1 - 1/2^n$. Compute $P(X_n \neq 0 \text{ i.o.})$.

Exercise 12 (**Probability theory/ Statistics 4**, 5 points)**.**
Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables taking values in $(0, \infty)$ such that the sequence $\log X_1, \log X_2, \ldots$ satisfies the assumptions for the law of large numbers. Prove that

$$(X_1 \cdots X_n)^{1/n} \xrightarrow[n\to\infty]{\text{almost surely}} \exp\left(\mathbb{E}\left[\log X_1\right]\right) .$$